London MathSpace - Diffie-Hellman key exchange

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Groups

Definition

A *Group* is a set G, with an operation

$$\cdot : G \times G \to G$$

$$(g, h) \mapsto g \cdot h =: gh$$

such that:

1. Associativity: $\forall g_1, g_2, g_3 \in G : (g_1 g_2) g_3 = g_1 (g_2 g_3)$
2. Existence of a neutral element:
   $\exists e \in G : \forall g \in G : eg = ge = g$
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By writing (for $n \in \mathbb{Z}$) $g^n = \underbrace{g \cdot \dots \cdot g}_{n \text{ times}}$ we get the usual identities

$g^{n+m} = g^n g^m$ and $(g^n)^m = (g^m)^n = g^{mn}$.

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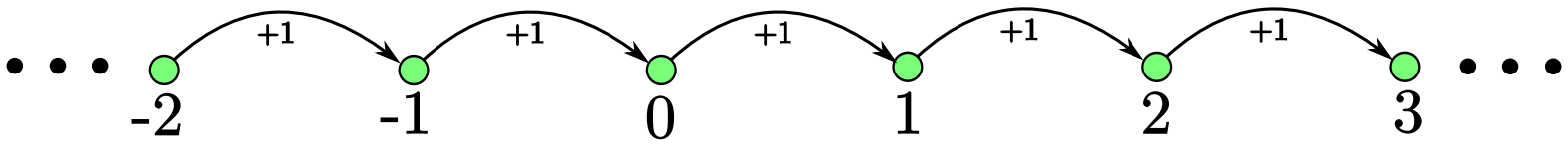
- $\mathbb{Z}/n\mathbb{Z}$, the integers modulo n, with operation of addition
- $(\mathbb{Z}/p\mathbb{Z})^{\times} := \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$ with p a prime number, with operation of multiplication

Cayley-Graph



Cayley-Graph of $\mathbb{Z}$.

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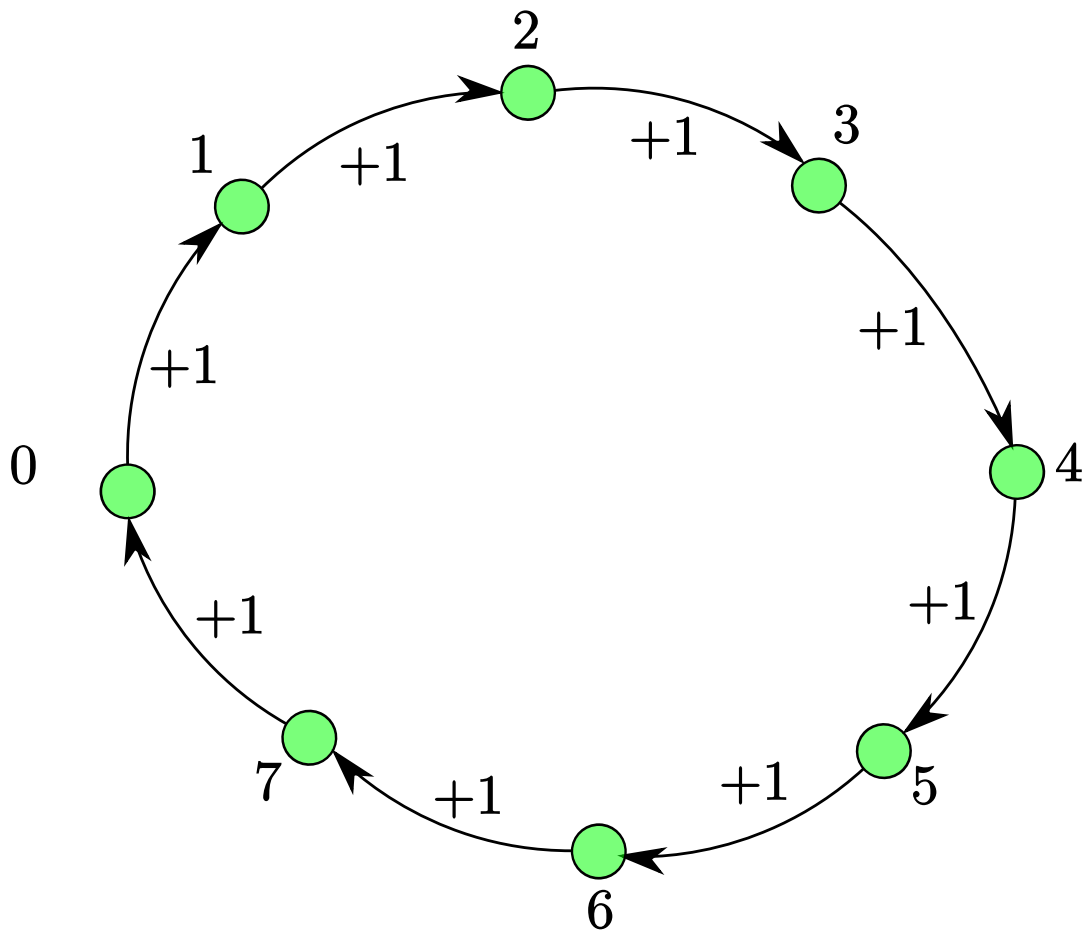

Cayley-Graph of $\mathbb{Z}/8\mathbb{Z}$.

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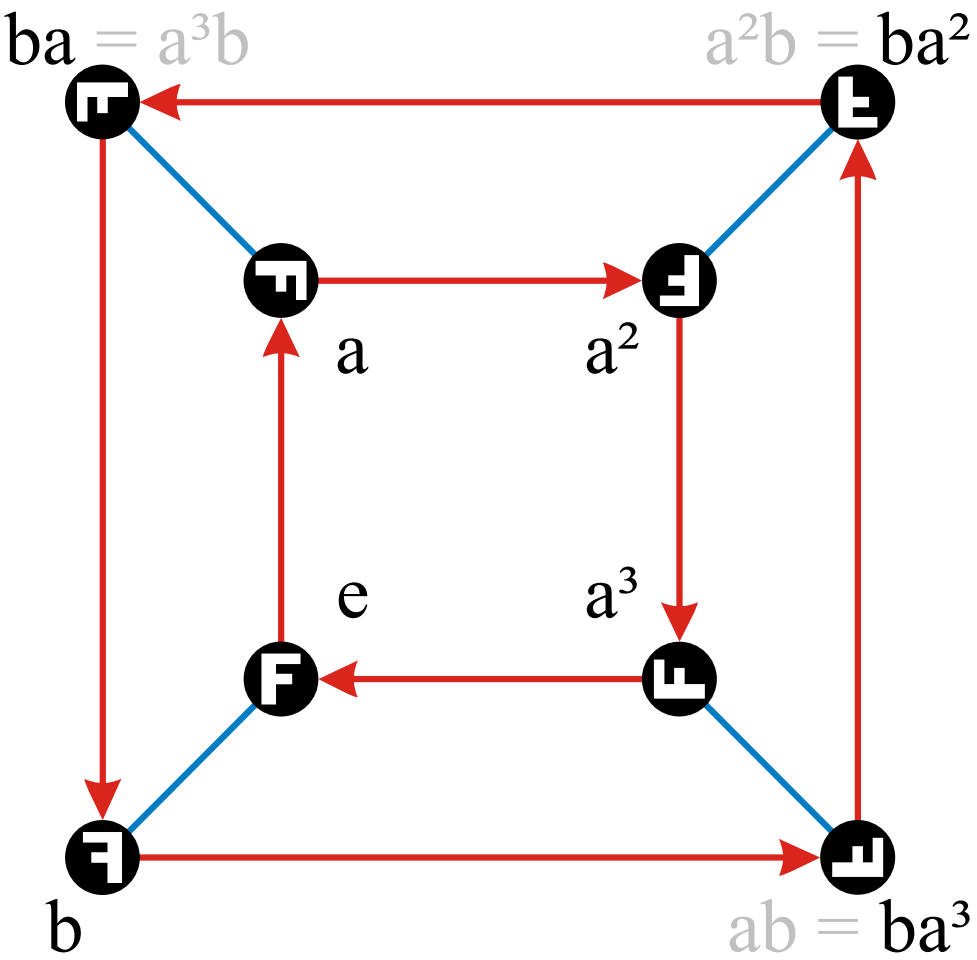

Cayley-Graph of D_4 (Symmetrien des Quadrats).

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- $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic.

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- A *Probabalistic Polynomial-Time* (PPT) Turing machine is a probabalistic turing machine with polynomial running-time.

Negligible functions

A function $\epsilon\colon \mathbb{N} \to \mathbb{N}$ is called *negligible*, if for every polynomial p, we have

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In the end, we have

$$k_A = y^a = (g^b)^a = g^{ba} = g^{ab} = (g^a)^b = x^b = k_B$$

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This is obviously stupid

Hardness assumptions

Let G be a finite cyclic group with generator g

Definition

The *computational discrete log problem* for the group G is the following: Find a PPT Turing machine A, that, given an element $h \in G$, outputs an $n \in \mathbb{Z}$, such that $\Pr[g^n = h]$ is non-negligible.

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If we can solve discrete log for G, we can also solve the DH-Problem for G.

Security of DH

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Candidate groups for having hard DH-Problems: $(\mathbb{Z}/p\mathbb{Z})^\times$ for large p, certain elliptic curve groups (ECC-DH).

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4. Alice computes $k_A = y^a = 64^{92} \equiv 256$. Bob computes $k_B = x^b = 241^{214} \equiv 256$.

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- At a future date, Eve gains acces to Alice's secret key. Since the communication is in the past, Eve can not compromise the verification. Thus confidentiality remains.
- Every *future* communication between Alice and Bob can be decyphered by Eve.

Questions and Discussion